Propositional Logic – Basic Concept

Adapted From:

NYU G22.2390-001, Propositional Logic
Stanford CS 103, Logic
教材《数理逻辑与集合论》1.1-1.3、2.3、2.4

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Propositional Logic （命题逻辑）

1.1 Propositional Logic

Step 0. Proof.

Step 1. Convert program into mathematical formula.

Step 2. Ask the computer to solve the formula.

1.2 First Order Logic

Step 1. Convert it into first logic formula.

Step 2. Ask the computer to solve the formula.

1.3 Interactive Proof

Step 1. Axiom system

Step 2. Ask the computer to check the invariants

Step 1.1 What is Propositional Logic?

Why We Need Propositional Logic?

Goal: Formalize the theorem, definitions and reasoning we use in our proofs.

Propositional Logic

What is a proposition (命题)***:***
A ***proposition*** is a statement that is, by itself, either true or false.

Examples

Theorem proved before are all propositions:

If n is an even integer, then n^2 is even.

If n^2 is an even integer, then n is even.

If n is an integer and n > 0, then $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Two queens problem has no solution.

Examples

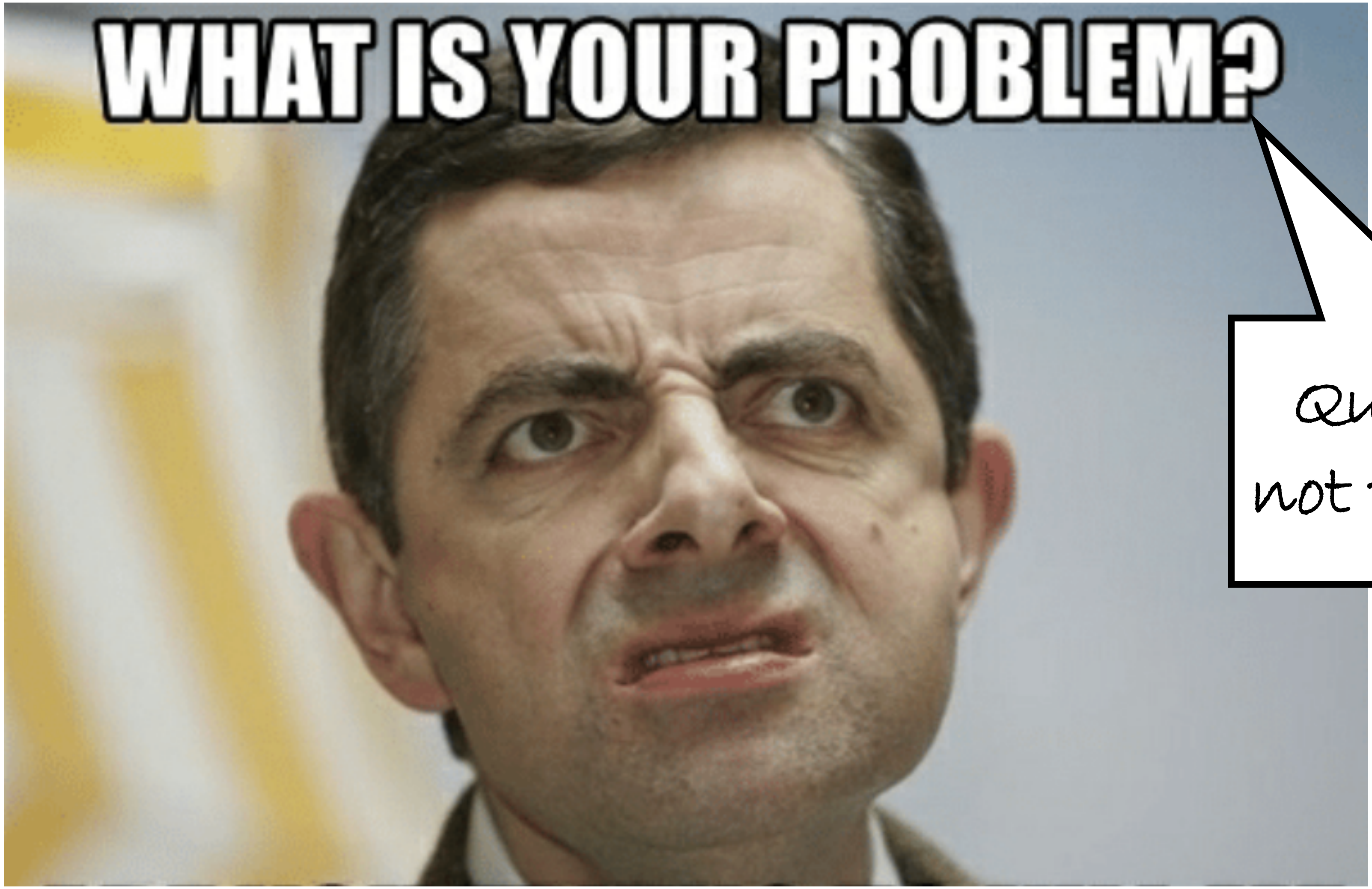
Tom is taller than Jerry

$2 + 2 = 4$

$N > 6$

Discrete mathematics is edible

3 is odd



Questions are not propositions.

Commands are not propositions.


Exclamatory sentence are not propositions.

Propositional Logic

Propositional logic (命题逻辑) is a mathematical system for reasoning about propositions and how they relate to one another.

Propositional Logic

Every formula in propositional logic consists of ***propositional variables*** combined via ***propositional connectives.***

- Each variable represents some proposition, such as "Tom is a cat" or "Jerry is a rat."
- Connectives encode how propositions are related, such as "Tom is a cat ***and*** Jerry is a rat."

Propositional Variables（命题变项）

Propositional variables are used to represent proposition (simple propositions).

Propositional variables are usually represented as upper-case letters

- For example, P and Q.
- We can use P to represent “Tom is taller than Jerry”

Every propositional variable has a truth value

- It is either true or false

Propositional Connectives（命题联结词）

Propositional Connectives are used to

- Connect propositional variables
- Express more complex meaning with existing propositions

P1: “Tom is a cat.”

P2: “Jerry is a rat.”

If Tom is a cat and Jerry is a rat, then Tom and Jerry are enemy.

P3: “Tom and Jerry are enemy.”

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Propositional Connectives are used to

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P1: "Tom is a cat."

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P3: "Tom and Jerry are enemy."

⇨

If Tom is a cat and Jerry is a rat, then Tom and Jerry are enemy.

Atomic Proposition (原子命题)

Propositions without any connectives are called atomic propositions or simple propositions（原子命题/简单命题）

P1: “Tom is a cat.”

P2: “Jerry is a rat.”

P3: “Tom and Jerry are enemy.”

Atomic Propositions

⇨

If Tom is a cat and Jerry is a rat, then Tom and Jerry are enemy.

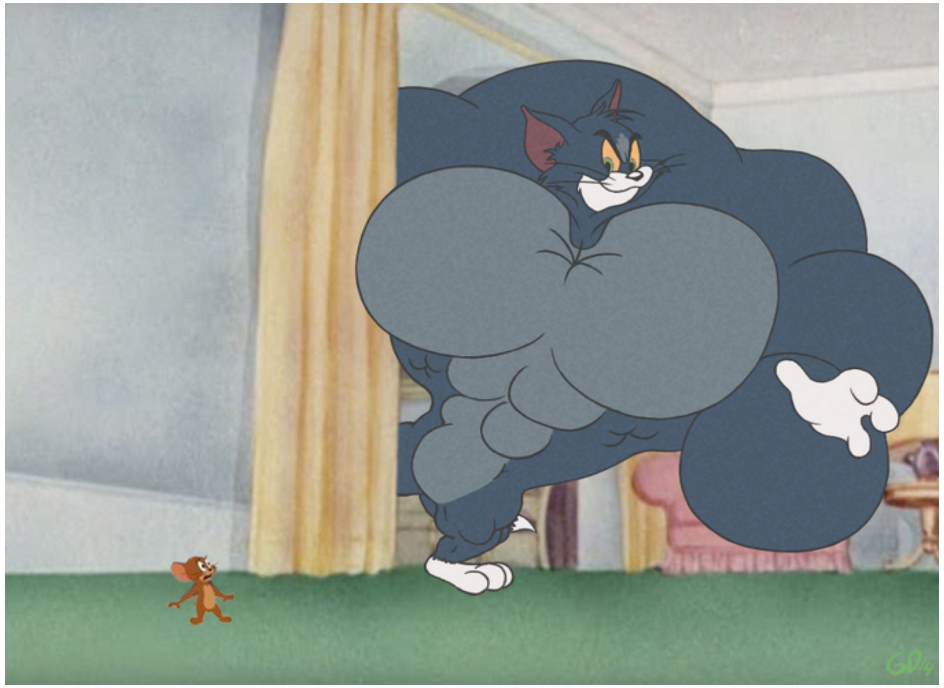
Proposition

NOT (Logical Negation)（否定词）

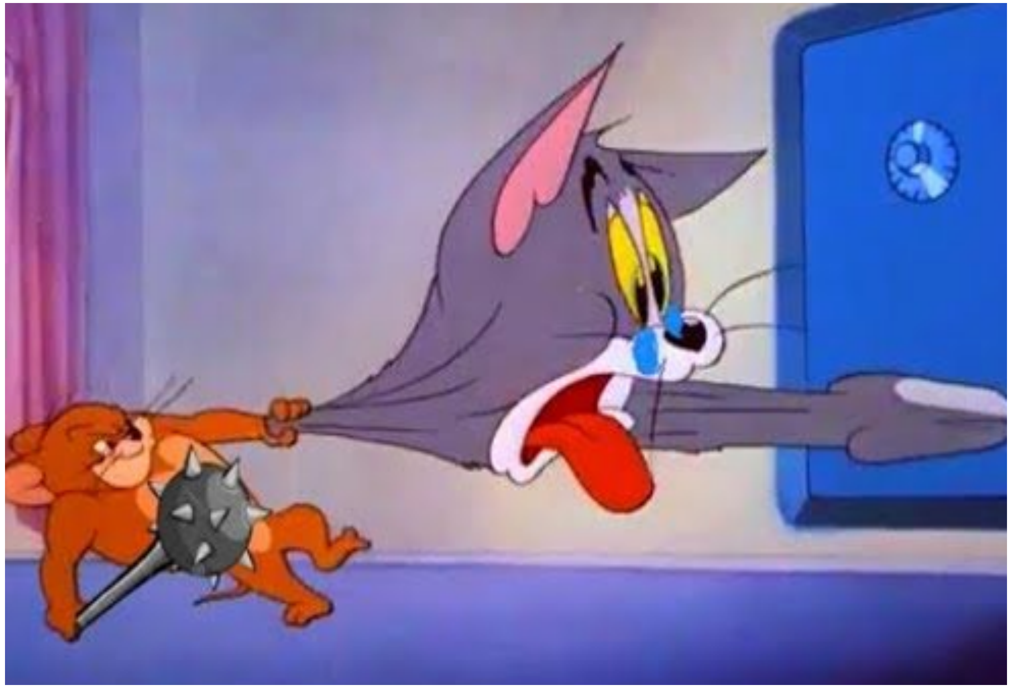
P : Tom is stronger than Jerry

$\neg P$: Tom is **NOT** stronger than Jerry

P



$\neg P$



Truth Table (真值表)

A truth table is a table

- showing the truth value of a propositional logic formula as a function of its inputs.
- describing the semantic of a propositional connective / formula.

The Semantic of “NOT”

Propositional connective

Propositional variable

Propositional formula

P	$\neg P$
T	F
F	T

Truth Table

$\neg \boldsymbol{P}$ always has a different truth value with P.

AND (Logical Conjunction)（合取词）

P： Tom is shaking hands with Jerry.

Q： Tom is shaking hands with Quacker.

$P \wedge Q$： Tom is shaking hands with Jerry **AND** Tom is shaking hands with Quacker.

$P \wedge Q$

The Semantic of "AND"

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

They are the same.

Truth Table

$\boldsymbol{P \wedge Q}$ is true if P is true and Q is true.

OR (Logical Disjunction)（析取词）

P : Krusty Krab has burger.

Q : Krusty Krab has pizza.

$P \vee Q$: Krusty Krab has burger **OR** Krusty Krab has pizza.

$P \vee Q$

The Semantic of “OR”

P	Q	$P \vee Q$	$Q \vee P$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

$\boldsymbol{P} \vee \boldsymbol{Q}$ is true if P is true or Q is true.

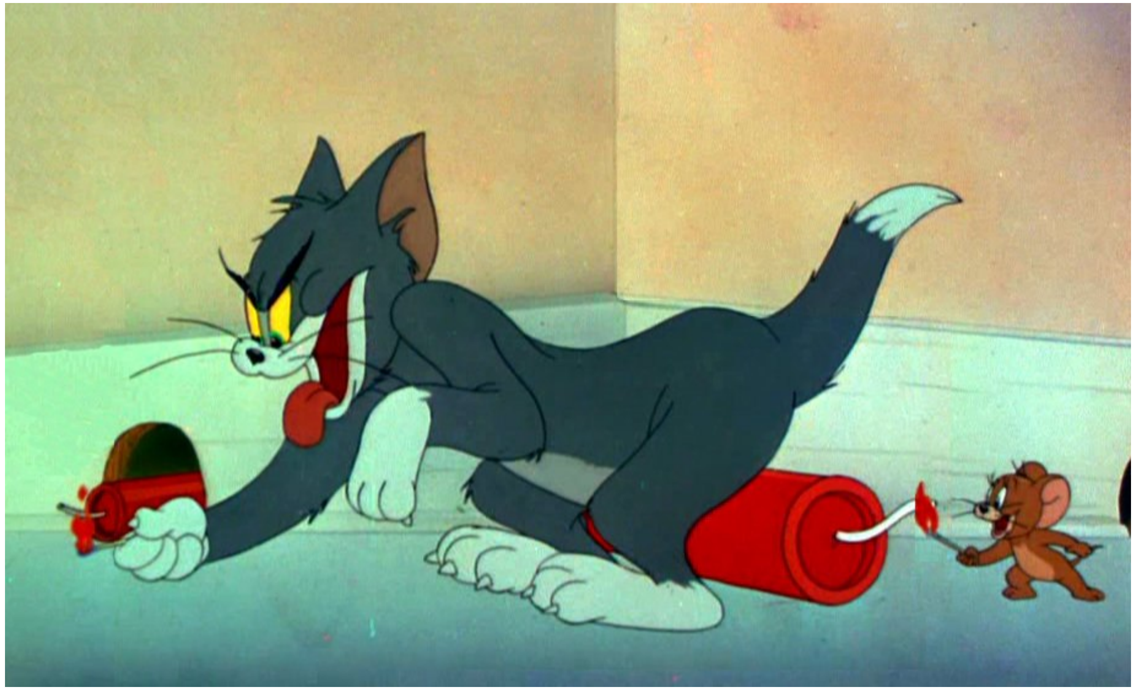
Implication（蕴涵词）

P：Jerry fires the firecracker.

Q：The firecracker will explode.

$P \rightarrow Q$：**If** Jerry fires the firecracker, **then (implies)** the firecracker will explode.

$P \rightarrow Q$



The Semantic of "Implication"

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

The implication is only false if P is true and Q isn't. You need to commit this to memory.

Biconditional Connective（双条件词）

P : SpongeBob is on the right of Patrick Star.

Q ：Patrick Star is on the left of SpongeBob.

$P \leftrightarrow Q$ ：SpongeBob is on the right of Patrick Star **if and only if** Patrick Star is on the left of SpongeBob.

$P \leftrightarrow Q$

The Semantic of "Biconditional Connective"

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

The $\boldsymbol{P} \leftrightarrow \boldsymbol{Q}$ is only true if P and Q have the same truth value.

The Semantic of “Biconditional Connective”

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

It seems $\boldsymbol{P} \leftrightarrow \boldsymbol{Q}$ means “P implies Q” and “Q implies P”. But can you prove it?

The Semantic of "Biconditional Connective"

P	Q	$P \leftrightarrow Q$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

They have the same truth values.

A Quick Recap

A propositional formula includes variables and connectives.

Propositional variables: P, Q …

Propositional connectives

Name	Symbol	#Variables	Example
NOT	$\neg$	1	$\neg P$
AND	$\land$	2	$P \land Q$
OR	$\lor$	2	$P \lor Q$
Implication	$\rightarrow$	2	$P \rightarrow Q$
Biconditional Connective	$\leftrightarrow$	2	$P \leftrightarrow Q$

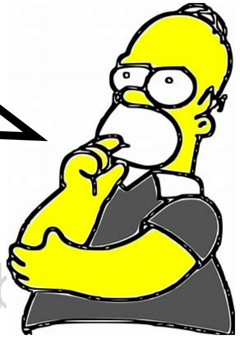
A Quick Recap

A p… …iables and connectives.

Pr…

Pr…

How to connect 3 or more variables?



Name	Symbol	#Variab…	Example
Im…			…Q
Biconditional Connective	↔	2	P ↔ Q

INTUITION

With multiple connectives.

$P \wedge Q \wedge (\wedge R)$

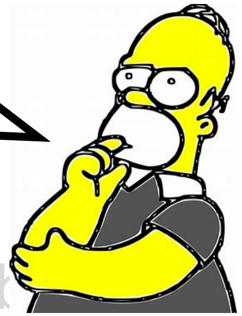
Well-Formed Formulas (合式公式)

A p... iables and connectives.

Pr...

Pr...

How to connect 3 or more variables legally?



Name	Symbol	#Variab...	Example
N...			P
			Q
			Q
Im...			Q
Biconditional Connective	↔	2	P ↔ Q

Well-Formed Formulas (合式公式)

If a formula is a well-formed formula (wff), then it is legal.

Well-Formed Formulas (合式公式)

INDUCTIVE DEFINITION of WFF

1). Every single proposition (symbol) is in WFF.

2). If A and B are WFF, so are $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$.

3). No expression is WFF unless forced by 1) or 2).

A Quick Recap

A propositional formula includes variables and connectives.

Propositional variables: P, Q …

Propositional connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

WFF: construct the legal propositional formula.

A Quick Recap

A propositional formula includes variables and connectives.

Propositional variables: P, Q …

Propositional connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

WFF: construct the legal propositional formula.

Why these are enough?